Chebyshev Polynomials

Recall that the error in polynomial interp. is given by

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

• Assume (for convenience) that the interpolation nodes are in $[-1,1]$. If $x \in [-1,1]$ then $\xi_x \in [-1,1]$, so

$$\max_{x \in [-1,1]} |f(x) - p(x)| \leq \frac{1}{(n+1)!} \max_{x \in [-1,1]} |f^{(n+1)}(\xi_x)| \cdot \max_{x \in [-1,1]} \left| \prod_{i=0}^{n} (x - x_i) \right|$$

Idea: Choose the nodes x_i to minimize this term!

Observe: $\prod_{i=0}^{n}(x-x_i)$ is a monic polynomial

coefficient of x^n is 1.

Theorem: If p is a monic polynomial of degree n then

$$\|p\|_\infty := \max_{x\in[-1,1]} |p(x)| \geq 2^{1-n} \quad (*)$$

To prove this, we will construct a polynomial that achieves the bound.

Chebyshev Polynomials:

2 Definitions (equivalent)

Recursive Definition:

$$\begin{cases} T_0(x)=1, \quad T_1(x)=x \\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n\geq 1 \end{cases}$$

$$\Rightarrow T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1 \quad \ldots$$

$$\vdots$$

Equivalent Definition / Theorem

$$T_n(x) = \cos(n \cos^{-1} x), \quad n \geq 0$$

Proof of equivalence:

Since $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\Rightarrow \cos(n+1)\theta = \cos\theta \cos n\theta - \sin\theta \sin n\theta$$

$$\cos(n-1)\theta = \cos\theta \cos n\theta + \sin\theta \sin n\theta$$

$$\Rightarrow \cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta \cos n\theta$$

plug in $\cos^{-1} x$

& define $f_n(x) = \cos(n \cos^{-1} x) \Rightarrow f_0(x) = 1, \ f_1(x) = x$

& $f_{n+1}(x) + f_{n-1}(x) = 2x f_n(x)$

$$\iff f_{n+1}(x) = 2x f_n(x) - f_{n-1}(x)$$

So $T_n = f_n \quad \forall n$ $\blacksquare$

Recall that we want to get an upper bound on $|p(x)|$, $x \in [-1,1]$ when p is monic.

Theorem: $\|p\|_\infty = \max\limits_{x\in[-1,1]} |p(x)| \geq 2^{1-n}$

proof: Suppose that $|p(x)| < 2^{1-n} \quad \forall x \in [-1,1]$

We want to get a contradiction.

Let $q(x) = 2^{1-n} T_n(x)$ & let $x_i = \cos\left(\frac{i\pi}{n}\right)$.

Observe that q is monic, degree n. Also:

$$q(x_i) = 2^{1-n}\cos\left(n \cdot \frac{i\pi}{n}\right) = 2^{1-n}(-1)^i$$

$$\text{So} \quad q(x_i)(-1)^i = 2^{1-n} \underset{\text{by our supposition}}{>} |p(x_i)| \geq (-1)^i p(x_i)$$

$$\Rightarrow (-1)^i \Big(\underbrace{\overset{\text{monic}}{q(x_i)} - \overset{\text{monic}}{p(x_i)}}_{\text{poly., degree} \leq n-1} \Big) > 0 \qquad \forall i \in \{0, \dots, n\}$$

(highest degree terms cancel)

So, we have a polynomial of degree $\leq n-1$ that changes sign $n+1$ times in $[-1,1]$ $\Rightarrow$ it has n-roots. Can't happen with degree $\leq n-1$

$\Rightarrow$ contradiction $\Rightarrow$ $\underline{|P_n(x)| \geq 2^{1-n}}$ $\boxed{\text{N}}$

OK, let's get back to the error in poly. interp.

$$\max_{x\in[-1,1]} |f(x)-p(x)| \leq$$

$$\frac{1}{(n+1)!} \max_{x\in[-1,1]} |f^{(n+1)}(x)| \max_{x\in[-1,1]} \underbrace{\left|\prod_{i=0}^{n}(x-x_i)\right|}_{\text{monic degree } n}$$

$\Rightarrow \geq 2^{-n}$

So, the best we ~~can~~ hope to do is 2^{-n}.

From the proof above, we want $\displaystyle\prod_{i=0}^{n}(x-x_i) = 2^{-n}T_{n+1}(x)$

with nodes $x_i = \cos\left(\frac{2i+1}{2n+2}\pi\right)$, $i=0,\dots,n$

So we proved :

Theorem

If x_i are the the roots of T_{n+1} :

$$|f(x) - P(x)| \leq \frac{2^{-n}}{(n+1)!} \max_{|t|\leq 1} |f^{(n+1)}(t)|$$